**Special examples where the treatment number is six**

This write-up is to start discussing some special optimal designs that were found and dissected these designs to get some better understandings of their properties.

To study the optimal designs that were found where the Phase 1 experiment is arranged in randomised block design (RBD), I would first compared the treatment average efficiency factors with the optimal design where the Phase 1 experiment is arranged in completely randomised design (CRD). This can point me to the sets of design parameters that may be harder come up with the better design.

The first observation is that treatment average efficiency factors from both cases are identical where the treatment numbers are 2, 3, 4, 5 and 7.

However, the treatment average efficiency factors are not the same only when the treatment numbers are 6 and 8, more specifically, under the cases where the number of the runs is not divisible by the treatment number. (4 and 8 tags for 6 treatments and only 4 tags for 8 treatments)

The case that I am going to look at here is where the design parameters are consists of of v = 6, r\_b = 3, r\_t = 2, n\_\gamma = 4 and n\_R = 15. Note that the number 15 is not divisible by 6. If the first phase experiment is RBD, then number of cage, n\_C = 3.

**The first phase is CRD**

The first phase theoretical ANOVA is expressed as follows

$ANOVA

DF e Ani

Between Ani

Trt 5 1 2

Residual 12 1 2

Within 18 1 0

$EF

Trt eff.Trt

Between Ani

Trt 6 1

Within

Based on the random effects table, all the treatment information is in the Between Animal stratum.

The following allocation of animals to runs and tags were found and expressed as matrix where the upper case letter denotes the animal ID,

[,1][,2][,3][,4]

[1,] "K" "D" "G" "N"

[2,] "D" "K" "N" "G"

[3,] "A" "P" "Q" "C"

[4,] "P" "A" "C" "Q"

[5,] "I" "H" "J" "L"

[6,] "H" "I" "L" "J"

[7,] "M" "R" "O" "B"

[8,] "R" "M" "B" "O"

[9,] "F" "F" "E" "E"

The following allocation of treatments to runs and tags were found and expressed as matrix where the lower case letter denotes each treatments,

[,1] [,2] [,3] [,4]

[1,] "e" "d" "a" "b"

[2,] "d" "e" "b" "a"

[3,] "a" "d" "e" "c"

[4,] "d" "a" "c" "e"

[5,] "c" "b" "d" "f"

[6,] "b" "c" "f" "d"

[7,] "a" "f" "c" "b"

[8,] "f" "a" "b" "c"

[9,] "f" "f" "e" "e"

The canonical efficiency factors, average efficiency factor and eigenvectors associated with the treatment effects in the Between Animals Within Runs stratum for this design can be expressed as follows

$can.eff

[1] 0.9166667 0.9166667 0.8888889 0.7500000 0.7500000

$ave.eff

[1] 0.8370323

$e.vec

[,1] [,2] [,3] [,4] [,5] [,6]

[1,] 6.207546e-01 -3.386204e-01 -0.4082483 0.40549902 -0.04729914 -0.4082483

[2,] 3.386204e-01 6.207546e-01 0.4082483 0.04729914 0.40549902 -0.4082483

[3,] -3.386204e-01 -6.207546e-01 0.4082483 0.04729914 0.40549902 -0.4082483

[4,] -6.207546e-01 3.386204e-01 -0.4082483 0.40549902 -0.04729914 -0.4082483

[5,] -1.948962e-15 2.914335e-15 0.4082483 -0.09459829 -0.81099805 -0.4082483

[6,] 2.792905e-15 2.220446e-16 -0.4082483 -0.81099805 0.09459829 -0.4082483

The second phase theoretical ANOVA is expressed as follows

DF e Ani Run

Between Run

Between Ani

Trt 4 1 2 4

Residual 4 1 0 4

Within

Between Ani

Tag 1 1 2 0

Trt 5 1 2 0

Residual 7 1 2 0

Residual

Tag 2 1 0 0

Residual 12 1 0 0

$EF

Tag Trt eff.Tag eff.Trt

Between Run

Between Ani

Trt 3/4 1/8

Residual

Within

Between Ani

Tag 9 2/3 1 1/9

Trt 7920/1577 1320/1577

Residual

Tag 9 1

Compared this ANOVA table of the first phase experiment, 5 DF associated with the animal effects are lost from the first phase experiment. The 4 DF of 5DF for the animals are now in the Between Runs stratum and 1 DF associated with tag effects is now in the Between Animals Within Runs stratum. In addition, the amount of treatment information remains for conducting the test is now 0.8370 from 100% of the first phase experiment. The five eigenvectors described are used as the treatment contrasts and fitted to the theoretical ANOVA table to study how the treatment information are separated for each treatment contrast and gives

$ANOVA

DF e Ani Run

Between Run

Between Ani

Trt.1 1 1 2 4

Trt.2 1 1 2 4

Trt.4 1 1 2 4

Trt.5 1 1 2 4

Residual 4 1 0 4

Within

Between Ani

Tag 1 1 2 0

Trt.1 1 1 2 0

Trt.2 1 1 2 0

Trt.3 1 1 2 0

Trt.4 1 1 2 0

Trt.5 1 1 2 0

Residual 7 1 2 0

Residual

Tag 2 1 0 0

Residual 12 1 0 0

eff.Tag eff.Trt.1 eff.Trt.2 eff.Trt.3 eff.Trt.4 eff.Trt.5

Between Run

Between Ani

Trt.1 1/12

Trt.2 1/12

Trt.4 1/4

Trt.5 1/4

Residual

Within

Between Ani

Tag 1 1/9

Trt.1 11/12

Trt.2 11/12

Trt.3 8/9

Trt.4 3/4

Trt.5 3/4

Residual

Tag 1

This theoretical ANOVA table shows the amount of the treatment information for each of the five orthogonal treatment contrasts. The treatment contrasts 1 and 2 both consist of 1/12 and 11/12 of the treatment information in Between Runs and Within Runs strata, respectively. The tag mean square in the Between Animals Within Runs stratum contains 1/9 of treatment information from the treatment contrast 3, which mean there is still 8/9 of the pure treatment information remaining. The treatment contrasts 4 and 5 both consist of 1/4 and 3/4 of the treatment information in Between Runs and Within Runs strata, respectively.

**The first phase is RBD**

The first phase theoretical ANOVA is expressed as follows

$ANOVA

DF e Cag:Ani Cag

Between Cag 2 1 2 12

Between Cag:Ani

Trt 5 1 2 0

Residual 10 1 2 0

Within 18 1 0 0

$EF

Trt eff.Trt

Between Cag

Between Cag:Ani

Trt 6 1

Within

Now all the treatment information is in the Between Animals Within Cages stratum. Note that there are 2 DF associated with the Between Cages stratum, so the DF for the residual MS for the Between Animals Within Cages stratum is now 10 compared to the DF of the residual MS for the Between Animals stratum were 12 for the CRD.

The following allocation of cages and animals to runs and tags were found and expressed as matrix where the upper case letter denotes the Cage ID and numeric numbers denotes the animal ID within each cage,

[,1] [,2] [,3] [,4]

[1,] "A1" "B4" "A5" "B6"

[2,] "B4" "A1" "B6" "A5"

[3,] "B3" "A6" "B1" "A2"

[4,] "A6" "B3" "A2" "B1"

[5,] "A4" "B2" "A3" "B5"

[6,] "B2" "A4" "B5" "A3"

[7,] "C5" "C1" "C2" "C4"

[8,] "C1" "C5" "C4" "C2"

[9,] "C3" "C3" "C6" "C6"

One initial observation is that the Cages is confounded with Runs, because Run 1, 2, 3, 4, 5 and 6 contain Cages A and B whereas Run 7, 8 and 9 contain only Cage C. Hence, the allocation of the cages to runs is disconnected. This can be observed in the cages incidence matrix with respect to runs and cage concurrence matrix with respected to runs, which can be expressed as

> (N = with(design.df, table(Cag,Run)))

Run

Cag 1 2 3 4 5 6 7 8 9

a 2 2 2 2 2 2 0 0 0

b 2 2 2 2 2 2 0 0 0

c 0 0 0 0 0 0 4 4 4

> N %\*% t(N)

Cag

Cag a b c

a 24 24 0

b 24 24 0

c 0 0 48

Note that since the number of cages is smaller than the tag number; hence, the allocation of cages to runs can never be binary. The following allocation of treatments to runs and tags were found and expressed as matrix,

[,1] [,2] [,3] [,4]

[1,] "d" "e" "c" "a"

[2,] "e" "d" "a" "c"

[3,] "a" "b" "f" "c"

[4,] "b" "a" "c" "f"

[5,] "f" "c" "b" "e"

[6,] "c" "f" "e" "b"

[7,] "b" "a" "d" "f"

[8,] "a" "b" "f" "d"

[9,] "e" "e" "d" "d"

The canonical efficiency factors and average efficiency factor associated with the treatment effects in the Between Animals Within Cages Within Runs stratum for this design can be expressed as follows

$can.eff

[1] 1.0000000 0.8943376 0.8888889 0.8333333 0.6056624

$ave.eff

[1] 0.8204481

$e.vec

[,1] [,2] [,3] [,4] [,5] [,6]

[1,] -0.5773503 0.4082483 -0.4082483 3.511783e-15 0.4082483 -0.4082483

[2,] -0.2886751 -0.5576775 0.4082483 -5.000000e-01 0.1494292 -0.4082483

[3,] 0.2886751 0.1494292 -0.4082483 -5.000000e-01 -0.5576775 -0.4082483

[4,] -0.2886751 0.1494292 0.4082483 5.000000e-01 -0.5576775 -0.4082483

[5,] 0.2886751 -0.5576775 -0.4082483 5.000000e-01 0.1494292 -0.4082483

[6,] 0.5773503 0.4082483 0.4082483 1.471046e-15 0.4082483 -0.4082483

Note that this average efficiency factor is slightly smaller than the previous case of 0.8370.

The trace of the treatment information matrix is 25.333 and the square of the treatment information matrix is 131.444.

The second phase theoretical ANOVA is expressed as follows,

$ANOVA

DF e Cag:Ani Cag Run

Between Run

Between Cag 1 1 2 12 4

Between Cag:Ani

Trt 3 1 2 0 4

Residual 4 1 0 0 4

Within

Between Cag 1 1 2 12 0

Between Cag:Ani

Tag 1 1 2 0 0

Trt 5 1 2 0 0

Residual 6 1 2 0 0

Residual

Tag 2 1 0 0 0

Residual 12 1 0 0 0

$EF

Tag Trt eff.Tag eff.Trt

Between Run

Between Cag

Between Cag:Ani

Trt 1 1/6

Residual

Within

Between Cag

Between Cag:Ani

Tag 9 2/3 1 1/9

Trt 15600/3169 2600/3169

Residual

Tag 9 1

Compared this ANOVA table to the previous one, 4 DF associated with the animal effects are lost from the first phase experiment. The 3 DF of 4DF for the animals are now in the Between Runs stratum and 1 DF associated with tag effects is now in the Between Animals Within Runs stratum. In addition, amount of treatment information remain for conducting the test is now 0.8204 from 100% of the first phase experiment, where the Phase 1 experiment is arranged in CRD, the amount of treatment information remain for conducting the test was 0.8370. The five eigenvectors described are used as the treatment contrasts and fitted to the theoretical ANOVA table and gives

$ANOVA

DF e Cag:Ani Cag Run

Between Run

Between Cag 1 1 2 12 4

Between Cag:Ani

Trt.2 1 1 2 0 4

Trt.4 1 1 2 0 4

Trt.5 1 1 2 0 4

Residual 4 1 0 0 4

Within

Between Cag 1 1 2 12 0

Between Cag:Ani

Tag 1 1 2 0 0

Trt.1 1 1 2 0 0

Trt.2 1 1 2 0 0

Trt.3 1 1 2 0 0

Trt.4 1 1 2 0 0

Trt.5 1 1 2 0 0

Residual 6 1 2 0 0

Residual

Tag 2 1 0 0 0

Residual 12 1 0 0 0

$EF

eff.Tag eff.Trt.1 eff.Trt.2 eff.Trt.3 eff.Trt.4 eff.Trt.5

Between Run

Between Cag

Between Cag:Ani

Trt.2 0.1057

Trt.4 1/6

Trt.5 0.3943

Residual

Within

Between Cag

Between Cag:Ani

Tag 1 1/9

Trt.1 1

Trt.2 0.8943

Trt.3 8/9

Trt.4 5/6

Trt.5 0.6057

Residual

Tag 1

This ANOVA table shows the amount of the treatment information for each of the five orthogonal treatment contrasts. The treatment contrast 1 has all its information in the Between Animals Within Cages Within Runs stratum. The tag mean square in the Between Animals Within Cages Within Runs stratum contains 1/9 of treatment information from the treatment contrast 3, which mean there is still 8/9 of the pure treatment information remaining. The treatment contrasts 2, 4 and 5 have 0.1057, 1/6 and 0.3943 of the treatment information in the Between Animals Within Cages Between Runs stratum, respectively. The harmonic mean of 0.1057, 1/6 and 0.3943 is 1/6; this confirms the previous theoretical ANOVA table has 1/6 of the treatment information in the Between Animals Within Cages Between Runs stratum. The amount pure treatment information for treatment contrasts 2, 4 and 5 are 0.8943, 5/6 and 0.6057.

For the above design, the treatment average efficiency factor can be improved by sacrificing one DF associated with the residual mean square in the Between Animals Within Cages Within Runs stratum. The better design can be expressed in the following allocation of cages and animals to runs and tags as matrix where the upper case letter denotes the Cage ID and numeric numbers denotes the animal ID within each cage,

[,1] [,2] [,3] [,4]

[1,] "B5" "C6" "A3" "C2"

[2,] "C6" "B5" "C2" "A3"

[3,] "B3" "C4" "C1" "A5"

[4,] "C4" "B3" "A5" "C1"

[5,] "B2" "C5" "B1" "A6"

[6,] "C5" "B2" "A6" "B1"

[7,] "A1" "A2" "B4" "C3"

[8,] "A2" "A1" "C3" "B4"

[9,] "A4" "A4" "B6" "B6"

Now the allocation of cages is connected with respect to runs, because each run always contains at least three different cages. This can be observed in the cages incidence matrix with respect to runs and cage concurrence matrix with respected to runs are

> (N = with(design.df, table(Cag,Run)))

Run

Cag 1 2 3 4 5 6 7 8 9

a 1 1 1 1 1 1 2 2 2

b 1 1 1 1 2 2 1 1 2

c 2 2 2 2 1 1 1 1 0

> N %\*% t(N)

Cag

Cag a b c

a 18 16 14

b 16 18 14

c 14 14 20

Note that for the previous case where the cage allocation to runs is disconnected; hence, the amount of the cage information in both Between Runs and Within Runs strata is 100%, but 2 DF associated with cage effects are split to one in the Between Runs stratum and one in the Within Runs stratum. For this case, the cage allocation to runs is connected, so there should be 2 DF associated with cages effects in both Between Runs and Within Runs strata. However, the amount of the cage information should be lesser than 100%. The canonical efficiency factors associated cages in the Between Runs stratum are

[1] 1/8 1/24

and in the Within Runs stratum are

[1] 23/24 7/8.

The following allocation of treatments to runs and tags were found and expressed as matrix,

[,1] [,2] [,3] [,4]

[1,] "b" "e" "d" "a"

[2,] "e" "b" "a" "d"

[3,] "e" "c" "b" "f"

[4,] "c" "e" "f" "b"

[5,] "a" "d" "c" "e"

[6,] "d" "a" "e" "c"

[7,] "f" "d" "b" "c"

[8,] "d" "f" "c" "b"

[9,] "f" "f" "a" "a"

The canonical efficiency factors, average efficiency factor and eigenvectors associated with the treatment effects in the Between Animals Within Cages Within Runs stratum for this design can be expressed as follows

$can.eff

[1] 0.9149129 0.9113076 0.8888889 0.7445074 0.7315496

$ave.eff

[1] 0.8298127

$e.vec

[,1] [,2] [,3] [,4] [,5] [,6]

[1,] 0.01063288 0.03316809 0.4082483 0.5772523 -0.5763967 -0.4082483

[2,] -0.49459876 0.48259018 -0.4082483 0.2978345 0.3169228 -0.4082483

[3,] 0.50523164 -0.51575827 -0.4082483 0.2794178 0.2594740 -0.4082483

[4,] 0.49459876 0.48259018 0.4082483 -0.2978345 0.3169228 -0.4082483

[5,] -0.01063288 0.03316809 -0.4082483 -0.5772523 -0.5763967 -0.4082483

[6,] -0.50523164 -0.51575827 0.4082483 -0.2794178 0.2594740 -0.4082483

Note that this average efficiency factor is slightly higher than the previous design of 0.8204, but it is still slightly smaller than the first case of 0.8370. In addition, the trace of the treatment information matrix is 25.147 and the square of the treatment information matrix is 127.696. Hence, this design is not an MS-optimal designs compared to the previous even though this design has the higher average efficiency factor.

The second phase theoretical ANOVA is expressed as follows,

$ANOVA

DF e Cag:Ani Cag Run

Between Run

Between Cag

Trt 2 1 2 1 4

Between Cag:Ani

Trt 2 1 2 0 4

Residual 4 1 0 0 4

Within

Between Cag

Trt 2 1 2 11 0

Between Cag:Ani

Tag 1 1 2 0 0

Trt 5 1 2 0 0

Residual 5 1 2 0 0

Residual

Tag 2 1 0 0 0

Residual 12 1 0 0 0

$EF

Tag Trt eff.Tag eff.Trt

Between Run

Between Cag

Trt 1 1/6

Between Cag:Ani

Trt 1 1/6

Residual

Within

Between Cag

Trt 1/15 1/90

Between Cag:Ani

Tag 9 2/3 1 1/9

Trt 78960/15859 13160/15859

Residual

Tag 9 1

For this design, the treatment mean squares can be seen in every stratum apart from the Within Animals Within Cages Within Runs stratum. The amount of treatment information remain for conducting the test is now 0.8298 which is higher than the previous design of 0.8204; however, the DF associated with the residual in the Between Animals Within Cages Within Runs stratum has reduced to five. The five eigenvectors described are used as first treatment contrasts and fitted to the theoretical ANOVA table and gives

$ANOVA

DF e Cag:Ani Cag Run

Between Run

Between Cag

Trt.1 1 1 2 1/2 4

Trt.2 1 1 2 3/2 4

Between Cag:Ani

Trt.1 1 1 2 0 4

Trt.2 1 1 2 0 4

Residual 4 1 0 0 4

Within

Between Cag

Trt.1 1 1 2 23/2 0

Trt.2 1 1 2 21/2 0

Between Cag:Ani

Tag 1 1 2 0 0

Trt.1 1 1 2 0 0

Trt.2 1 1 2 0 0

Trt.3 1 1 2 0 0

Trt.4 1 1 2 0 0

Trt.5 1 1 2 0 0

Residual 5 1 2 0 0

Residual

Tag 2 1 0 0 0

Residual 12 1 0 0 0

$EF

eff.Tag eff.Trt.1 eff.Trt.2 eff.Trt.3 eff.Trt.4 eff.Trt.5

Between Run

Between Cag

Trt.1 0.03904 0.1276

Trt.2 0.03366 0.133

Between Cag:Ani

Trt.1 0.04435 0.1223

Trt.2 0.05022 0.1164

Residual

Within

Between Cag

Trt.1 0.001697 0.005549

Trt.2 0.0048 0.019

Between Cag:Ani

Tag 1 1/9

Trt.1 0.9149

Trt.2 0.9113

Trt.3 8/9

Trt.4 0.7445

Trt.5 0.7315

Residual

Tag 1

This theatrical ANOVA table shows the amount of the treatment information for each of the five orthogonal treatment contrasts. From the previous table, the amount of the treatment information in the Between Cages Between Runs stratum is 1/6, which can be deduced from the sum of 0.03904 and 0.1276 and the sum of 0.03366 and 0.133. Similarly, the amount of the treatment information in the Between Animals Within Cages Between Runs stratum is 1/6, which can be deduced from the sum of 0.04435 and 0.1223 and the sum of 0.05022 and 0.1164. The treatment contrast 1 in the Between Cages Within Runs consists 0.001697 and 0.005549 of the treatment information and their sum is 1/138. The treatment contrast 2 in the Between Cages Within Runs consists 0.0048 and 0.019 of the treatment information and their sum is 1/42. The harmonic mean of 1/138 and 1/42 is 1/90. This theatrical ANOVA table can be transform to the theatrical ANOVA table of the first case by combining the all treatment information of Cages and Animals Within Cages mean squares.

In conclusion, this write up shows that design with one lesser DF associated with the residual mean square for conducting the test of the treatment effects can also have higher treatment average efficiency factor. This must be considered when the under the cases of same set of design parameters ignoring the cages, the treatment average efficiency factor obtained when the Phase 1 experiment is arranged in the RBD is lower than when the Phase 1 experiment is arranged in the CRD.